

Bayes rule, generalized discord and nonextensive thermodynamics

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Generalized measurement-induced correlations are derived by taking the Bayes rule as fundamental principle. Proven non-negative and limited from above, the resulting quantifiers have shown to admit operational interpretation in terms of the efficiency of quantum and classical demons in allowing for the extraction of generalized work from a heat bath. The link with discord is established by adopting the q -entropy as entropic principle. This allows us to reproduce, within a one-parameter formalism, both the entropic and the geometric measures of discord. Besides offering a unified view of several correlations in terms of a Bayesian principle and its connection with thermodynamics, our approach unveils a bridge to the nonextensive statistical mechanics.

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Correlations occupy a prominent place in our present models of Nature. Acquisition of empiric *information* from the world is typically mediated by the fingerprints left by these resources in physical pointers. The information stored in the apparatus—a physical memory which necessarily gets an increase of *entropy*—can eventually be used to extract *work* from a heat bath. The whole process is such that we can learn about and benefit from the laws of Nature without violating any of them [1, 2].

Entanglement figures in this context as a class of correlations which cannot be prepared by local operations and classical communication (LOCC) [3]. It permeates many physical arenas, from foundational phenomena, such as nonlocality [4] and decoherence [5], to the interplay with the quantum information science [6, 7]. About a decade ago, however, a novel class of quantum correlations has been identified which manifests even in separable states. Since then, the so called *quantum discord* [8, 9], along with some of its variants [10–13], has revealed its importance in many different contexts (see Ref. [14] for a recent review on the subject). One of the versions of this information theoretic measure, here named *entropic discord* and stated as

$$D_E(\rho) = \min_{\{\Pi_y\}} \left(S_1(\Pi_Y[\rho_Y]) + \sum_y p_y S_1(\rho_{X|y}) \right) - S_1(\rho), \quad (1)$$

has been linked with the difference between the efficiency of quantum and classical demons in extracting thermodynamic work from a heat bath [14, 15]. Above, $\rho_{X|y} = \text{Tr}_Y(\Pi_y \rho) / p_y$ is the conditional density matrix, $p_y = \text{Tr}(\Pi_y \rho)$ is the probability of the outcome y , $\Pi_Y[\rho] = \sum_y \Pi_y \rho \Pi_y$, $\Pi_y = |y\rangle\langle y|$ is a von Neumann projector associated with a discrete observable Y , and S_1 is the von Neumann entropy. Alternative measures have been constructed via geometric principles. They have shown to be of great potential for both theoretical [14] and experimental investigations [16, 17]. In its seminal version, the *geometric discord* [10] reads

$$D_G(\rho) := \min_{\Pi_Y} \|\rho - \Pi_Y[\rho]\|^2, \quad (2)$$

where $\|\rho\|^2 := \text{Tr}(\rho^\dagger \rho)$ is the square norm in the Hilbert-Schmidt space.

The quest for fine understanding of these and other measurement-induced correlations [18, 19] and their eventual connections with thermodynamics is a formidable current problem in quantum physics [2, 15, 20]. In particular, it is legitimate to ask whether the available measures capture different quantum correlations or are just different mathematical expressions of the same resource. Whichever the case, it is insightful to look for unifying principles which reveal the actual substance of a set of measures. This paper aims to give some contributions in this direction. First, a generalized measure of the quantumness of correlations is derived from primitive notions of the theory of probabilities. Second, an operational interpretation is given which relates our measure with the extra amount of information a quantum demon can provide from a state in comparison with a classical demon. Finally, by adopting a specific entropic principle, we obtain a unified view for the entropic and geometric measures of discord while establishing a link with the nonextensive thermodynamics.

Bayesian correlations.—At the core of classical theory of probabilities is the notion of conditional probability, $\wp_{X|y} := \wp_{X,y} / \wp_y$. According to the Bayesian interpretation, $\wp_{X|y}$ refers to the knowledge available about a random variable X *after* a given outcome y has been measured for a random variable Y . On the other hand, $\wp_{X,y}$ and \wp_y respectively denote the probability of $X \cap (Y=y)$ and the probability of the outcome y , with no reference to measurements. They can be regarded as knowledge *prior* to the inference process. From $\wp_{X|y} = (\wp_{X,y} / \wp_X \wp_y) \wp_X$ one sees how prior knowledge \wp_X is updated under a measurement, as long as X and Y are dependent events. Generically, we can write

$$\sum_y f(\wp_y \wp_{X|y}) = \sum_y f(\wp_{X,y}), \quad (3)$$

for some function f . This formula—hereafter referred to as *Bayes rule*—highlights the essence of

the classical insensitivity to local measurements, as it equals a measurement-dependent term (the l.h.s) with a measurement-independent one (the r.h.s.). A quantum mechanical analog of this formula can be tried as follows. Let us take $\rho_{X|y}$ as the *updated* reduced state. From its spectral decomposition one proves that $[\text{Tr}_Y(\Pi_y \rho)]^q = \text{Tr}_Y[(\Pi_y \rho \Pi_y)^q]$, $q \geq 1 \in \mathbb{R}$. Then, assuming that f is *analytic* and noting that the operators $\Pi_y \rho \Pi_y$ have mutually orthogonal supports we obtain

$$\text{Tr}_X \sum_y f(p_y \rho_{X|y}) = \text{Tr} f(\Pi_Y[\rho]), \quad (4)$$

where $\Pi_Y[\rho] = \sum_y \Pi_y \rho \Pi_y$. In contrast with Eq. (3), in the above the r.h.s. makes clear reference to projective measurements. Nevertheless, this is not so for all states. In fact, if $\sigma' = \Pi_Y[\sigma]$, then $\text{Tr} f(\Pi_Y[\sigma']) = \text{Tr} f(\sigma')$. For any other state, the Bayesian principle does not apply. This motivates us to define a quantifier of the *least deviation from Bayes rule* induced by a local measurement,

$$\Delta B(\rho) := \min_{\Pi_Y} \text{Tr} \left(f(\rho) - f(\Pi_Y[\rho]) \right) \geq 0. \quad (5)$$

Non-negativity is readily demonstrated for any *convex* f by means of the generalized Klein's inequality [7] along with the relation $\text{Tr}[\rho(\Pi_Y[\rho])^{q-1}] = \text{Tr}[(\Pi_Y[\rho])^q]$, $\forall q \geq 1 \in \mathbb{R}$. A byproduct of the proof is the upper bound

$$\Delta B^{ub}(\rho) = \min_{\Pi_Y} \text{Tr} \left\{ (\rho - \Pi_Y[\rho]) f'(\rho) \right\}. \quad (6)$$

By construction, it is clear that $\Delta B=0$ iff the state does not change under local measurements, i.e., $\Pi_Y[\rho] = \rho$. Now, sensitivity to local disturbance, non-negativity, and mathematical structure intimately related with the trace distance, are symptoms of quantifiers of measurement-induced correlations. Thus, the Bayesian measure (5) emerges as a possible precursor of discord and other measures of the quantumness of correlations. In what follows we show that this is indeed the case. Firstly, however, we assess the physical meaning of ΔB .

Operational interpretation.—So far we have assumed that f is analytic and convex. In contraposition to entropy, usually taken as a *concave* measure of ignorance, it is natural to associate f with *information* (\mathcal{I}). Following Ref. [21] we propose that

$$\text{Tr} f(\rho) = \mathcal{S}_{max} - \mathcal{S}(\rho) \equiv \mathcal{I}(\rho), \quad (7)$$

where $\mathcal{S}(\rho)$ denotes an arbitrary entropic measure and \mathcal{S}_{max} is a constant used to set $\mathcal{I} = 0$ for maximally mixed states. Now, given that $\Delta B = \mathcal{I}(\rho) - \max_{\Pi_Y} \mathcal{I}(\Pi_Y[\rho])$ we can elaborate on traditional demonic protocols [15].

Charlie wants to bargain with Maxwell's demons in order to get information about a given state ρ . The available demonic beings, however, never reveal the outcome of a measurement, but only the observable measured. Charlie then examines two classes of spirits. He

knows that a *classical demon* can only perform local operations on the system. After correlating the system X with an eigenbasis $\{|y\rangle\}$ of his physical apparatus, the demon could read off the pointer Y , say at position y , and predict the state $\rho_{X|y}|y\rangle\langle y|$. Without information about the outcome, Charlie would have to average over all possibilities, thus accessing only a partial amount $\mathcal{I}(\sum_y p_y \rho_{X|y}|y\rangle\langle y|)$ of information. Being lucky enough, Charlie might invoke a demon which always chooses the optimal basis, in which case he would benefit from $\max_{\Pi_Y} \mathcal{I}(\Pi_Y[\rho])$. On the other hand, a *quantum demon* can perform measurements in global bases corresponding to observables that commute with the state of the system XY . Having learned about the measured observable, Charlie could infer ρ and thus accumulate an amount $\mathcal{I}(\rho)$ of information. Charlie then concludes that quantum demons are more effective than classical ones as the former can offer an amount ΔB of extra information. (In order not to lose his soul—nor violate thermodynamic laws—Charlie needs to erase the memory of the demon after completion of the service.)

The link with thermodynamics can be established by noting that the informational content of ρ allows Charlie to draw from a heat bath of temperature \mathcal{T} an amount of work $\mathcal{W}(\rho) = k\mathcal{T}\mathcal{I}(\rho)$ [21]. This step, however, is constrained to the existence of a thermodynamic structure for generalized quantities $\{k, \mathcal{T}, \mathcal{S}, \mathcal{W}\}$ which preserves the form of the usual laws derived from the Boltzmann-Gibbs-von Neumann entropy S_1 . Assuming that this is the case, we get

$$\mathcal{W}(\rho) - \max_{\Pi_Y} \mathcal{W}(\Pi_Y[\rho]) = k\mathcal{T}\Delta B(\rho). \quad (8)$$

Ultimately, this result points out that the extra work a quantum demon allows Charlie to extract from a heat bath of temperature \mathcal{T} , by use of a state ρ , is determined by how much the (unread) local measurement Π_Y violates the Bayes rule.

Relation to quantum discord.—The connection with other measures of quantumness of correlations is established by specializing the entropic principle in Eq. (7). In order to keep some degree of generality, we consider the unified (q, r) -entropy [22, 23]. The class arising for $r = 1$ will be of particular interest here. It refers to the Tsallis q -entropy [24],

$$S_q(\rho) := \frac{1 - \text{Tr} \rho^q}{q - 1} \quad (q > 0), \quad (9)$$

which is concave whenever $q \geq 1$ and reduces to the von Neumann entropy (logarithm given in natural base) as $q \rightarrow 1$. In addition, note that it recovers the linear entropy $S_2 = 1 - \text{Tr} \rho^2$ as $q = 2$. We are now in position to define a generalized discord, $D_q(\rho) := [\Delta B(\rho)]_{S_q}$, as a specialization of the Bayesian measure induced by the Tsallis entropy. From Eqs. (5)-(7) we then obtain

$$D_q(\rho) = \min_{\Pi_Y} \left(S_q(\Pi_Y[\rho]) - S_q(\rho) \right), \quad (10)$$

with upper bound

$$D_q^{ub}(\rho) = \min_{\Pi_Y} q \left(\frac{\text{Tr} \rho^q - \text{Tr}(\Pi_Y[\rho] \rho^{q-1})}{q-1} \right). \quad (11)$$

Extensions to other (q, r) -entropies can be carried on straightforwardly. The positivity of D_q implies that the q -entropy cannot decrease under local von Neumann measurements, i.e., $S_q(\Pi_Y[\rho]) \geq S_q(\rho)$. This is an important advantage to other formulations [25].

The relation of the q -discord (10) with D_E and D_G can be readily verified. From $\Pi_Y[\rho] = \sum_y p_y \rho_{X|y} |y\rangle\langle y|$ it is easy to show that $\text{Tr}[(\Pi_Y[\rho])^q] = \sum_y p_y^q \text{Tr}_X[(\rho_{X|y})^q]$. Then, since $\sum_y p_y^q = \text{Tr}_Y[(\Pi_Y[\rho_Y])^q]$, we obtain

$$S_q(\Pi_Y[\rho]) = S_q(\Pi_Y[\rho_Y]) + \sum_y p_y^q S_q(\rho_{X|y}),$$

which is the q -version of the *joint entropy theorem* [7]. Equation (10) is then rewritten as

$$D_q(\rho) = \min_{\Pi_Y} \left(S_q(\Pi_Y[\rho_Y]) + \sum_y p_y^q S_q(\rho_{X|y}) \right) - S_q(\rho), \quad (12)$$

where $S_q(X|Y) = \sum_y p_y^q S_q(\rho_{X|y})$ is the Tsallis conditional entropy. It is now obvious by Eqs. (1) and (12) that $D_{q \rightarrow 1}(\rho) = D_E(\rho)$. The link with the geometric discord is less apparent but easily shown as well. The natural guess is to look at $q = 2$, as in this case we have a direct link between entropy and geometry, i.e., $1 - S_2(\rho) = \|\rho\|^2$. Using previously derived relations we get

$$\begin{aligned} \|\rho - \Pi_Y[\rho]\|^2 &= \text{Tr} \rho^2 + \text{Tr}[(\Pi_Y[\rho])^2] - 2\text{Tr}[(\Pi_Y[\rho])^2] \\ &= S_2(\Pi_Y[\rho]) - S_2(\rho), \end{aligned}$$

which proves that $D_{q=2}(\rho) = D_G(\rho)$. Geometric discord turns out to be, therefore, a deviation of Bayes rule modeled by the linear entropy.

The results above provide a unified view of discord as Bayesian correlations induced by q -entropies. It is worth noticing that further measures can be conceived by taking the symmetric form of Bayes rule, $\wp_{x,y} = \wp_y \wp_{x|y} = \wp_x \wp_{y|x}$. Following our previous procedure one may readily suggest $\delta B := \min_{\Pi} \text{Tr}[f(\rho) - f(\Pi[\rho])]$, with $\Pi[\rho] = \sum_{x,y} \Pi_{xy} \rho \Pi_{xy}$ and $\Pi_{xy} = \Pi_x \otimes \Pi_y$, as a measure of joint disturbance. It would be interesting to assess how δB compare to other global quantifiers, such as *AMID* [18] and *quantum deficit* [21].

Examples.—For any pure state $\rho = |\psi\rangle\langle\psi|$ one can take $|\psi_x\rangle = \langle y|\psi\rangle / \sqrt{p_y}$ to show that $\text{Tr}_X \langle y|\rho|y\rangle^q = \langle y|\rho_Y|y\rangle^q$. It follows that $\text{Tr}[(\Pi_Y[\rho])^q] = \text{Tr}_Y[(\Pi_Y[\rho_Y])^q] \leq \text{Tr}_Y \rho_Y^q$, the inequality deriving from $S_q(\Pi[\rho]) \geq S_q(\rho)$ [23]. Noting that $\text{Tr} \rho^q = 1$ we then obtain

$$D_q(|\psi\rangle) = S_q(\rho_Y) \leq \frac{q}{q-1} S_2(\rho_Y). \quad (13)$$

The upper bound was computed via Eq. (11). Supported by Ref. [26] the above expression extends to $q \geq 1$ the result according to which discord reduces to entanglement

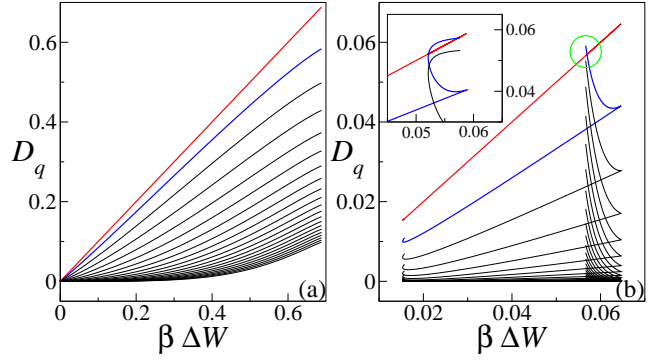


FIG. 1. Parametric plot of the q -discord versus the excess of (dimensionless) thermodynamic work $D_1 = \beta \Delta W$ offered by quantum demons by use of (a) the Werner state $\vec{c} = -v(1, 1, 1)$ and (b) the uv -state $\vec{c} = (u, v, \frac{u-v}{2})$, with $u = 1/3$, as a function of $v \in [0, u + \frac{2}{3}]$. From the top to the bottom, $q = 1 + n/2$ with $n = 0$ (red), 1 (blue), ..., 20. A similar plot is shown in the inset for $u = 0.32$ and $n = 0, 1, 2$. The uv -state is entangled if $(2-u)/3 < v < u - 2/3$.

for pure states. In particular, for a maximally entangled state, $|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle|i\rangle$, direct calculations yield

$$D_q(|\Psi\rangle) = \frac{d^{q-1} - 1}{(q-1)d^{q-1}} \leq \frac{q(d-1)}{d(q-1)},$$

from which one may readily show that $D_1(|\Psi\rangle) = \log d$ and $D_2(|\Psi\rangle) = 1 - \frac{1}{d}$, as expected. As a second example, we consider a two-qubit state with maximum marginals, $\varrho_{\vec{c}} = \frac{1}{4}(\mathbb{1} + \sum_{i=1}^3 c_i \sigma_i^X \otimes \sigma_i^Y)$, $c_i \in \mathbb{R}$. Using the spectral decomposition $\rho = \sum_i \lambda_i |i\rangle\langle i|$, where $\lambda_i(c_j) \in [0, 1]$, with ordering $\lambda_i \geq \lambda_j$ for $i > j$, and a well-known maximization algorithm [19] we obtain

$$\begin{aligned} D_q(\varrho_{\vec{c}}) &= \frac{1}{q-1} \left[\sum_{i=1}^4 \lambda_i^q - \frac{(1+c)^q + (1-c)^q}{2^{2q-1}} \right], \\ D_q^{ub}(\varrho_{\vec{c}}) &= \frac{q}{q-1} \left[\sum_{i=1}^4 \left(\lambda_i - \frac{1}{4} \right) \lambda_i^{q-1} - \frac{c_3}{4} \Lambda \right], \end{aligned}$$

where $c = \max\{|c_1|, |c_2|, |c_3|\}$ and $\Lambda = \lambda_4^{q-1} + \lambda_3^{q-1} - (\lambda_2^{q-1} + \lambda_1^{q-1})$. The limits $q \rightarrow 1, 2$ lead to the known results [14]. It is interesting to note that as q varies the entropic discord D_1 continuously deforms into the geometric discord D_2 and goes beyond, until reaching $D_\infty = 0$. This general property of D_q is illustrated in Fig. 1 for two specializations of $\varrho(\vec{c})$. Concerning the ordering of D_q , there is not a typical scenario. For instance, Fig. 1-(a) shows that $D_q(\rho) > D_q(\sigma)$ whenever $D_1(\rho) > D_1(\sigma)$, whereas in Fig. 1-(b) an inversion of this ordering is verified around $D_1 = 0.06$.

On the physics of “q”.—Although not universally accepted, the conceptual framework determined by the Tsallis entropy S_q and its underlying physical implications is supported by a substantial literature. In particular, considerable effort has been endeavored to generalize

the laws of thermodynamics (quantum counterparts included) [27–29] to arbitrary values of q . In parallel, a significant number of complex systems have been identified whose properties *cannot* be suitably described by the celebrated Boltzmann-Gibbs (BG) statistical mechanics, in which cases the Tsallis nonextensive statistical mechanics emerges as a well succeeded generalization (see [30] and references therein). Nevertheless, to the best of our knowledge, the precise (uncontroversial) physical meaning of the parameter q remains as a challenging question in the field of the nonextensive statistical mechanics.

As far as our results are concerned, two positions can be taken. On one hand, we can assume that S_q , and therefore D_q , are nothing but mathematical deformations induced by q . According to this view, no further physical content should be added to these generalized correlations, as they would be just different mimics of the same resources. If, on the other hand, the aforementioned context is not neglected—and we believe it should not—then we are invited to ascribe broader significance to the q -discord. According to this position, this measure consists in a natural generalization of its embryonic version D_1 to the context of complex systems [30], where the underlying microscopic dynamics dictates the appropriate value of q . The geometric discord D_2 , for example, would no longer be *an alternative* to D_1 , but *the* proper measure for the physics defined by $q = 2$.

Figure 1-(b) illustrates one of the subtleties of an eventual nonextensive theory of discord. For $q = 1.5$ unusual behavior is observed, namely, $D_{1.5}$ may become greater than the dimensionless work $D_1 = \beta W$. Operationally, this means that in this regime some states may offer optimal informational content. The question then immediately arises about what kind of system can generate such favorable conditions. It should be noted, however, that information gain, $D_q > D_1$, does not necessarily imply work gain, $\Delta \mathcal{W}_q > \Delta W$, because the very generalized notions of work \mathcal{W}_q and thermal energy $k_q \mathcal{T}_q$, along with pertinent thermodynamic relations, still wait to be formulated for arbitrary q [27–29]. This suggests an interesting research program on the relation of measurement-induced correlations and nonextensive statistical mechanics.

Conclusion.—We have derived a family of quantifiers for measurement-induced correlations by taking Bayesian rule as fundamental principle. The resulting measures have been shown to admit operational interpretation: they indicate by how much quantum demons are more efficient than classical ones in providing information about a given state. More importantly, this reveals a fundamental relation between (deviations from) Bayes rule and thermodynamics. In addition, by choosing a particular entropic principle, we have defined a generalized discord which reproduces the entropic and geometric measures by a proper adjustment of a single dimensionless parameter. Besides offering a unified framework for several

measurement-induced correlations, our approach opens the venue for challenging explorations of the information-work duet in the field of nonextensive thermodynamics.

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